Mclaurin [233 marks]

1. [Maximum mark: 19] 26M.1.AHL.TZ3.12

Consider $y = \tan\left(1 - \mathrm{e}^{3x}\right)$.

(a) Show that $\frac{\mathrm{d}y}{\mathrm{d}x} = -3\mathrm{e}^{3x}\left(1 + y^2\right)$. [4]

Markscheme

METHOD 1

attempt to use the chain rule ***M1***

$\frac{\mathrm{d}y}{\mathrm{d}x} = \sec^2\left(1 - \mathrm{e}^{3x}\right) \times \left(-3\mathrm{e}^{3x}\right)$ ***A1A1***

Note: Award ***A1*** for $\sec^2\left(1 - \mathrm{e}^{3x}\right)$ and ***A1*** for $-3\mathrm{e}^{3x}$.

$\frac{\mathrm{d}y}{\mathrm{d}x} = -3\mathrm{e}^{3x}\left(1 + \tan^2\left(1 - \mathrm{e}^{3x}\right)\right)$ ***A1***

$\frac{\mathrm{d}y}{\mathrm{d}x} = -3\mathrm{e}^{3x}\left(1 + y^2\right)$ ***AG***

METHOD 2

attempt to take $\arctan$ on both sides ***M1***

$\arctan y = 1 - \mathrm{e}^{3x}$

attempt to use implicit differentiation ***M1***

$\frac{1}{1+y^2}\frac{\mathrm{d}y}{\mathrm{d}x} = -3\mathrm{e}^{3x}$ ***A1A1***

Note: Award ***A1*** for $\frac{1}{1+y^2}\frac{\mathrm{d}y}{\mathrm{d}x}$ and ***A1*** for $-3\mathrm{e}^{3x}$.

$\frac{\mathrm{d}y}{\mathrm{d}x} = -3\mathrm{e}^{3x}\left(1 + y^2\right)$ ***AG***

[4 marks]

(b) Show that $\frac{\mathrm{d}^2y}{\mathrm{d}x^2} = -9$, when $x = 0$. [5]

Markscheme

use of product rule to differentiate $-3\mathrm{e}^{3x}\left(1 + y^2\right)$ ***(M1)***

use of implicit differentiation for $\left(1+y^2\right)$ ***(M1)***

all terms correct ***A1***

$\frac{\mathrm{d}^2y}{\mathrm{d}x^2} = -9\mathrm{e}^{3x}\left(1+y^2\right) - 3\mathrm{e}^{3x}\left(2y\frac{\mathrm{d}y}{\mathrm{d}x}\right)$

substitute $x = 0$ to find y or $\frac{\mathrm{d}y}{\mathrm{d}x}$

$y = 0$ **OR** $\frac{\mathrm{d}y}{\mathrm{d}x} = -3$ (seen anywhere) ***(A1)***

correct substitution into $\frac{\mathrm{d}^2y}{\mathrm{d}x^2}$ ***A1***

$\frac{\mathrm{d}^2y}{\mathrm{d}x^2} = -9\left(1+0^2\right) - 3(0)$

$\frac{\mathrm{d}^2y}{\mathrm{d}x^2} = -9$ ***AG***

[5 marks]

The function f is defined by $f(x) = \tan\left(1-\mathrm{e}^{3x}\right)$, where $-0.2 \leq x \leq 0.2$.

(c) Find the Maclaurin series for $f(x)$, up to and including the term in x^2. [2]

Markscheme

attempt to use Maclaurin series formula ***(M1)***

$f(x) = 0 + x(-3) + \frac{x^2}{2!}(-9)$

$f(x) = -3x - \frac{9}{2}x^2$ ***A1***

[2 marks]

(d) Hence, or otherwise, find $\lim_{x\to 0}\frac{\tan\left(1-\mathrm{e}^{3x}\right)}{x}$. [3]

Markscheme

METHOD 1

use of $\tan\left(1-\mathrm{e}^{3x}\right) = -3x - \frac{9}{2}x^2 - \cdots$ ***(M1)***

$\lim_{x\to 0}\frac{\tan\left(1-\mathrm{e}^{3x}\right)}{x} = \lim_{x\to 0}\frac{-3x-\frac{9}{2}x^2}{x}$ or $\lim_{x\to 0}\left(-3-\frac{9}{2}x\right)$ ***(A1)***

$= -3$ ***A1***

METHOD 2

use of L'Hopital's rule ***(M1)***

$\lim_{x\to 0}\frac{\tan\left(1-e^{3x}\right)}{x}=\lim_{x\to 0}\frac{\sec^2\left(1-e^{3x}\right)\times\left(-3e^{3x}\right)}{1}\left(=\lim_{x\to 0}\left(\frac{\left(1+y^2\right)\times\left(-3e^{3x}\right)}{1}\right)\right)$ ***A1***

Note: This ***A1*** line may be seen in part (b).

$=-3$ ***A1***

METHOD 3 (may be seen after part (e))

recognize $\lim_{x\to 0}\frac{\tan\left(1-e^{3x}\right)}{x}=\lim_{x\to 0}\frac{1}{a+bx}$ ***(M1)(A1)***

$=-3$ ***A1***

[3 marks]

The answer to part (c) is equal to the first two non-zero terms in the series expansion of $\frac{x}{a+bx}$, where $a,\ b\in\mathbb{Q}$.

(e) Find the value of a and the value of b. [5]

Markscheme

METHOD 1

use of Maclaurin series or extension of binomial theorem, $n\in\mathbb{Q}$ to expand $(a+bx)^{-1}$ ***(M1)***

$g\prime(x)=\frac{a}{(a+bx)^2}\left(\Rightarrow g\prime(0)=\frac{1}{a}\right)$ OR $\frac{x}{a}\left(1+(-1)\left(\frac{b}{a}x\right)+\cdots\right)$

$=\frac{x}{a}-\frac{b}{a^2}x^2$ ***A1***

Note: Award ***M1A0*** for one correct term.

comparing coefficients of x or x^2 ***(M1)***

$a=-\frac{1}{3}$ ***A1***

$b=\frac{1}{2}$ ***A1***

METHOD 2

recognition of infinite geometric series OR attempt to find common ratio ***(M1)***

$r=\frac{-\frac{9}{2}x^2}{-3x}$

common ratio $r=\frac{3}{2}x$ ***A1***

substituting $u_1 = -3x$ and $r = \frac{3}{2}x$ into S_∞ formula (and equating to $\frac{x}{a+bx}$) ***(M1)***

$S_\infty = \frac{-3x}{1-\frac{3}{2}x} \left(= \frac{x}{-\frac{1}{3}+\frac{1}{2}x}\right)$

$a = -\frac{1}{3}$ ***A1***

$b = \frac{1}{2}$ ***A1***

[5 marks]

2. [Maximum mark: 7] 25N.1.AHL.TZ1.5

The first four terms of the Maclaurin series expansion of $(1-x)^{-4}$ are

$1 + ax + bx^2 + 20x^3, a, b \in \mathbb{Z}^+$.

(a.i) Show that $a = 4$. [2]

Markscheme

METHOD 1

$f'(x) = 4(1-x)^{-5}$ ***A1***

substitution of $x = 0$ into derivative ***M1***

$4(1-0)^{-5}$ **OR** $f'(0) = 4$

$a = 4$ ***AG***

Note: Do not award ***M1*** for $1 + 4x$ seen without clear evidence of a substitution.

METHOD 2

attempt to create binomial expansion to at least two terms ***M1***

$1 + \binom{-4}{1}x + \left(\binom{-4}{2}(x)^2 + \ldots\right)$

correct expansion

$1 + (-4)(-x) + \left(\frac{(-4)(-5)}{2}(-x)^2 + \ldots\right)$ ***A1***

$a = 4$ ***AG***

Note: Do not award ***A1*** for $1 + 4x$ seen without clear evidence of correct working.

[2 marks]

(a.ii) Find the value of b. [3]

Markscheme

METHOD 1

$f''(x) = (-5) \times (4)(1-x)^6 \times (-1), 20(1-x)^{-6}$ ***(A1)***

$(f''(0) =)20$ ***(A1)***

$\frac{20}{2!}$

$b = 10$ ***A1***

METHOD 2

$\frac{(-4)(-5)}{2!}(-x)^2$ (may be seen in their expansion in (i)) ***(A1)***

$\frac{20}{2}$ ***(A1)***

$b = 10$ ***A1***

[3 marks]

A car was purchased four years ago. The car depreciated in value by 10% each year. The value of the car today is $\$1000$.

(b) Using the results of part (a), estimate the value of the car four years ago. [2]

Markscheme

attempt to substitute 0.1 into the expansion ***(M1)***

1.52

1520 (dollars) ***A1***

[2 marks]

3. [Maximum mark: 18] 25N.1.AHL.TZ1.10

Consider the expression $\frac{1}{1+x^2}$, where $|x| < 1$.

(a.i) Show that the first four terms in the binomial expansion of $\frac{1}{1+x^2}$ are $1 - x^2 + x^4 - x^6$. [3]

Markscheme

attempt to create binomial expansion to four terms with $n = -1$ ***(M1)***

$1 + \binom{-1}{1}x^2 + \binom{-1}{2}\left(x^2\right)^2 + \binom{-1}{3}\left(x^2\right)^3 + \ldots$

$1 - x^2 + \frac{(-1)(-2)}{2}\left(x^2\right)^2 + \frac{(-1)(-2)(-3)}{6}\left(x^2\right)^3$ ***A1A1***

Note: Award ***A1*** for the third term and ***A1*** for the fourth term.

$1 - x^2 + x^4 - x^6$ ***AG***

[3 marks]

(a.ii) Hence, find an approximation for $\int \frac{1}{1+x^2}\mathrm{d}x$ up to and including the term in x^7. [2]

Markscheme

$x - \frac{1}{3}x^3 + \frac{1}{5}x^5 - \frac{1}{7}x^7\ (+C)$ ***A2***

Note: Award ***A1A0*** for any three correct terms.

[2 marks]

Now consider the expression $\frac{1}{\sqrt{1-x^2}}$, where $|x| < 1$.

(b.i) Expand $\frac{1}{\sqrt{1-x^2}}$ up to and including the term in x^4. [4]

Markscheme

attempt to create binomial expansion the term in x^4 with $n = -\frac{1}{2}$ ***(M1)***

$1 + \binom{-\frac{1}{2}}{1}x^2 + \binom{-\frac{1}{2}}{2}\left(x^2\right)^2 + \cdots$

$1 - \frac{1}{2}\left(-x^2\right) + \frac{\left(-\frac{1}{2}\right)\left(-\frac{3}{2}\right)}{2!}\left(-x^2\right)^2$ ***(A1)(A1)***

Note: Award ***(A1)*** for the first two terms and ***(A1)*** for the third term.

$1 + \frac{1}{2}x^2 + \frac{3}{8}x^4$ **A1**

[4 marks]

(b.ii) Hence, find a polynomial expression for $\int \frac{1}{\sqrt{1-x^2}}\mathrm{d}x$ up to and including the term in x^5. [1]

Markscheme

$x + \frac{1}{6}x^3 + \frac{3}{40}x^5\ (+C)$ **A1**

[1 mark]

The expression $\frac{25}{48} + \frac{k}{1280}$, where $k \in \mathbb{Z}^+$, can be used to approximate $\arcsin\frac{1}{2}$.

(c) Use the result from part (b)(ii) to find the value of k. [4]

Markscheme

$\arcsin x = \int \frac{\mathrm{d}x}{\sqrt{1-x^2}}$ (seen anywhere) ***(A1)***

attempt to substitute $x = \frac{1}{2}$ into their antiderivative ***(M1)***

$\left(\arcsin\frac{1}{2} \approx\right) \frac{1}{2} + \frac{1}{6}\left(\frac{1}{2}\right)^3 + \frac{3}{40}\left(\frac{1}{2}\right)^5, \frac{1}{2} + \frac{1}{48} + \frac{3}{1280}$ **A1**

$k = 3$ (accept $\frac{25}{48} + \frac{3}{1280}$) **A1**

[4 marks]

(d) Hence, find an approximation for π in the form $\frac{p}{640}$ where $p \in \mathbb{Z}^+$. [4]

Markscheme

$\arcsin\left(\frac{1}{2}\right) = \frac{\pi}{6}$ (seen anywhere) ***(A1)***

equating their value of $\arcsin\left(\frac{1}{2}\right)$ to their expression from (c) ***(M1)***

$\frac{\pi}{6} \approx \frac{25}{48} + \frac{3}{1280}, \pi \approx 6 \times \left(\frac{25}{48} + \frac{3}{1280}\right)$

$\pi \approx \frac{25}{8} + \frac{18}{1280}, \frac{4000}{1280} + \frac{18}{1280}$ ***(A1)***

$\pi \approx \frac{2009}{640}$ (accept $p = 2009$) **A1**

Note: Condone use of "$=$" instead of "$\approx$".

[4 marks]

4. [Maximum mark: 19] 25M.1.AHL.TZ1.12

Consider the family of functions $f_n(x) = \cos^n x$, where $x \in \mathbb{R}$ and $n \in \mathbb{N}$.

(a) By writing $\cos^n x$ as $\cos^{n-1} x \cos x$, show that $\int \cos^n x \, \mathrm{d}x = \cos^{n-1} x \sin x + (n-1)\int \cos^{n-2} x \, \mathrm{d}x - (n-1)\int \cos^n x \, \mathrm{d}x$ for $n > 1$. [4]

Markscheme

attempt to use integration by parts on $f_n(x) = \cos^{n-1} x \cos x$ ***M1***

$u = \cos^{n-1} x$, $\frac{\mathrm{d}v}{\mathrm{d}x} = \cos x$

$\frac{\mathrm{d}u}{\mathrm{d}x} = -(n-1)\cos^{n-2} x \sin x$, $v = \sin x$

$\int f_n(x) \, \mathrm{d}x = \cos^{n-1} x \sin x + \int (n-1)\cos^{n-2} x \sin^2 x \, \mathrm{d}x$ ***A1A1***

Note: ***A1*** for each term with correct signs

$= \cos^{n-1} x \sin x + (n-1)\int \cos^{n-2} x\left(1 - \cos^2 x\right) \mathrm{d}x$ ***A1***

$= \cos^{n-1} x \sin x + (n-1)\int \cos^{n-2} x \, \mathrm{d}x - (n-1)\int \cos^n x \, \mathrm{d}x$ ***AG***

[4 marks]

(b) Hence, show that $\int f_n(x) \, \mathrm{d}x = \frac{1}{n}\cos^{n-1} x \sin x + \frac{n-1}{n}\int f_{n-2}(x) \, \mathrm{d}x$ for $n > 1$. [2]

Markscheme

attempt to rearrange the equation given in part (a) to collect terms in $\int f_n(x) \, \mathrm{d}x$ or $\int \cos^n x \, \mathrm{d}x$ ***M1***

$\int f_n(x) \, \mathrm{d}x + (n-1)\int f_n(x) \, \mathrm{d}x = \cos^{n-1} x \sin x + (n-1)\int f_{n-2}(x) \, \mathrm{d}x$ **OR**
$\int \cos^n x \, \mathrm{d}x + (n-1)\int \cos^n x \, \mathrm{d}x = \cos^{n-1} x \sin x + (n-1)\int \cos^{n-2} x \, \mathrm{d}x$

$n\int f_n(x) \, \mathrm{d}x = \cos^{n-1} x \sin x + (n-1)\int f_{n-2}(x) \, \mathrm{d}x$ **OR**
$n\int \cos^n x \, \mathrm{d}x = \cos^{n-1} x \sin x + (n-1)\int \cos^{n-2} x \, \mathrm{d}x$ ***A1***

$\int f_n(x) \, \mathrm{d}x = \frac{1}{n}\cos^{n-1} x \sin x + \frac{n-1}{n}\int f_{n-2}(x) \, \mathrm{d}x$ ***AG***

[2 marks]

(c) Hence, find an expression for $\int \cos^4 x \ \mathrm{d}x$, giving your answer in the form $p\cos^3 x \sin x + q\cos x \sin x + rx + c$ where $p,\ q,\ r \in \mathbb{Q}^+$. [4]

Markscheme

attempt to use equation from part (a) to reduce the power of $\cos x$ ***(M1)***

$\int \cos^4 x \ \mathrm{d}x = \frac{1}{4}\cos^3 x \sin x + \frac{3}{4}\int \cos^2 x \ \mathrm{d}x$

EITHER

attempt to use equation from part (a) again to reduce the power of $\cos x$ ***(M1)***

$\int \cos^4 x \ \mathrm{d}x = \frac{1}{4}\cos^3 x \sin x + \frac{3}{4}\left(\frac{1}{2}\cos x \sin x + \frac{1}{2}\int 1 \ \mathrm{d}x\right)$ ***(A1)***

OR

attempt to use double angle formula to rewrite $\cos^2 x$ in terms of $\cos 2x$ ***(M1)***

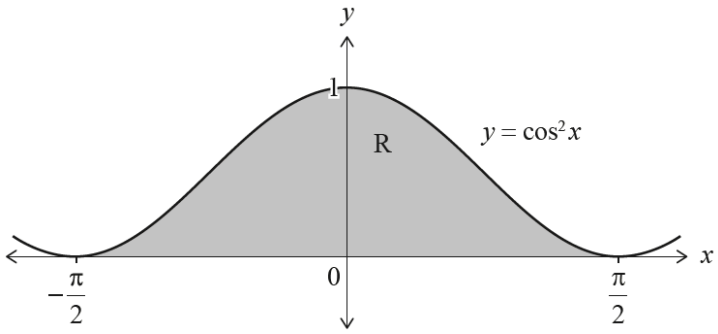
$\int \cos^4 x \ \mathrm{d}x = \frac{1}{4}\cos^3 x \sin x + \frac{3}{4}\int\left(\frac{\cos 2x+1}{2}\right) \mathrm{d}x$ ***(A1)***

THEN

$\frac{1}{4}\cos^3 x \sin x + \frac{3}{8}\cos x \sin x + \frac{3}{8}x + c$ ***A1***

[4 marks]

The region R is enclosed by the graph of $y = \cos^2 x$ and the x-axis where $-\frac{\pi}{2} \leq x \leq \frac{\pi}{2}$, as shown in the following diagram.



The region R is rotated by 2π radians around the x-axis to form a solid of revolution.

(d) Find the volume of the solid. [4]

Markscheme

attempt to use the formula for volume of revolution using π and $\left(\cos^2 x\right)^2$ ***(M1)***

volume $= \int \pi\left(\cos^2 \mathrm{x}\right)^2 \mathrm{d\ x}$

volume $= \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \pi\left(\cos^2 x\right)^2 \mathrm{d}\ x$ **OR** $2\int_{0}^{\frac{\pi}{2}} \pi\left(\cos^2 x\right)^2 \mathrm{d}\ x$ ***(A1)***

Note: Condone omission $\mathrm{d}\ x$ for the ***A1***.

attempt to substitute correct limits into their (c) and subtract ***(M1)***

$\pi\left(\frac{1}{4}\cos^3 \frac{\pi}{2}\sin \frac{\pi}{2} + \frac{3}{8}\cos \frac{\pi}{2}\sin \frac{\pi}{2} + \frac{3}{8}\left(\frac{\pi}{2}\right)\right) - \pi\left(\frac{1}{4}\cos^3 \left(-\frac{\pi}{2}\right)\sin\left(-\frac{\pi}{2}\right) + \frac{3}{8}\cos \left(-\frac{\pi}{2}\right)\sin \left(-\frac{\pi}{2}\right) + \frac{3}{8}\left(-\frac{\pi}{2}\right)\right)$

$= \frac{3\pi^2}{8}$ ***A1***

Note: ***FT*** marks may be awarded for a final answer of $r\pi^2$ based on non-zero values of p, q and r.

[4 marks]

(e.i) Find the Maclaurin series of $f_n(x)$ up to the term in x^2. [3]

Markscheme

METHOD 1

attempt to raise Maclaurin expansion for $\cos x$ to the power of n ***(M1)***

$f_n(x) = (\cos x)^n = \left(1 - \frac{x^2}{2} + \ldots\right)^n$

$= 1 - \frac{nx^2}{2} + \ldots$ ***A2***

METHOD 2

attempt to differentiate $f_n(x)$ twice ***(M1)***

$f_n\prime(x) = -n\cos^{n-1} x \sin x,\ f_n\prime\prime(x) = -n\cos^n x + n(n-1)\cos^{n-2} x \sin^2 x$ ***A1***

$f_n(0) = 1,\ f_n\prime(0) = 0,\ f_n\prime\prime(0) = -n$

$f_n(x) = 1 - \frac{nx^2}{2} + \ldots$ ***A1***

[3 marks]

(e.ii) Hence or otherwise, find $\lim_{x \to 0} \frac{f_n(x) - 1}{x^2}$ in terms of n. [2]

Markscheme

METHOD 1

attempt to use Maclaurin expansion for $f_n(x)$ ***(M1)***

$\lim_{x \to 0} \frac{f_n(x) - 1}{x^2} = \lim_{x \to 0} \frac{-\frac{nx^2}{2} + \ldots}{x^2} \left(= \lim_{x \to 0} \left(-\frac{n}{2} + \text{ powers of } x \right) \right)$

$= -\frac{n}{2}$ ***A1***

METHOD 2

attempt to use l'Hôpital's rule twice on $\frac{\cos^n(x) - 1}{x^2}$ ***(M1)***

$\lim_{x \to 0} \frac{\cos^n(x) - 1}{x^2} = \lim_{x \to 0} \frac{-n \cos^{n-1} x \sin x}{2x} = \lim_{x \to 0} \frac{n(n-1) \cos^{n-2} x \sin^2 x - n \cos^n x}{2}$

$= -\frac{n}{2}$ ***A1***

Note: Do not award ***FT*** marks for an expression that does not involve n.

[2 marks]

5. [Maximum mark: 17] 24N.1.AHL.TZ0.11

A curve is given by the equation $y = \frac{e^{2x} - 1}{e^{2x} + 1}$, $x \in \mathbb{R}$.

(a) By applying l'Hôpital's rule or otherwise, show that $\lim_{x \to \infty} \left(\frac{e^{2x} - 1}{e^{2x} + 1} \right) = 1$. [2]

Markscheme

EITHER

attempt to use l'Hôpital's ***(M1)***

$\lim_{x \to \infty} \left(\frac{e^{2x} - 1}{e^{2x} + 1} \right) = \lim_{x \to \infty} \frac{2e^{2x}}{2e^{2x}}$ ***A1***

OR

attempt to divide each term by e^{2x} ***(M1)***

$\lim_{x\to\infty}\left(\frac{e^{2x}-1}{e^{2x}+1}\right)=\lim_{x\to\infty}\left(\frac{1-e^{-2x}}{1+e^{-2x}}\right)$ ***A1***

THEN

$=1$ ***AG***

[2 marks]

(b.i) Show that $\frac{dy}{dx}=\frac{4e^{2x}}{(e^{2x}+1)^2}$. [3]

Markscheme

attempt to use quotient rule or product rule ***M1***

$\frac{dy}{dx}=\frac{(e^{2x}+1)2e^{2x}-(e^{2x}-1)2e^{2x}}{(e^{2x}+1)^2}$ ***A1A1***

Note: Award ***A1*** for first term in numerator, ***A1*** for second term in numerator. If denominator is incorrect award ***A1A0***.

$=\frac{4e^{2x}}{(e^{2x}+1)^2}$ ***AG***

[3 marks]

(b.ii) Hence, show that $1-y^2=\frac{dy}{dx}$. [3]

Markscheme

attempt to substitute for y and express as a single fraction ***M1***

$1-y^2=1-\frac{(e^{2x}-1)^2}{(e^{2x}+1)^2}$

$=\frac{(e^{2x}+1)^2-(e^{2x}-1)^2}{(e^{2x}+1)^2}$ or equivalent ***(A1)***

$=\frac{e^{4x}+2e^{2x}+1-(e^{4x}-2e^{2x}+1)}{(e^{2x}+1)^2}$ OR $\frac{2e^{2x}\times 2}{(e^{2x}+1)^2}$ ***A1***

$=\frac{4e^{2x}}{(e^{2x}+1)^2}$ ***AG***

[3 marks]

(c.i) By using implicit differentiation and the result in part (b)(ii), show that $\frac{d^2y}{dx^2}=2y^3-2y$. [3]

Markscheme

attempt to use implicit differentiation **M1**

$\left(\frac{\mathrm{d}^2y}{\mathrm{d}x^2} =\right) -2y\frac{\mathrm{d}y}{\mathrm{d}x}$ **A1**

$= -2y(1-y^2)$ **A1**

$= 2y^3 - 2y$ **AG**

[3 marks]

(c.ii) Hence, find an expression for $\frac{\mathrm{d}^3y}{\mathrm{d}x^3}$ in terms of y. [2]

Markscheme

$\left(\frac{\mathrm{d}^3y}{\mathrm{d}x^3} =\right) \left(6y^2-2\right)\frac{\mathrm{d}y}{\mathrm{d}x}$ **A1**

$= \left(6y^2-2\right)\left(1-y^2\right) \quad \left(= 8y^2 - 6y^4 - 2\right)$ **A1**

[2 marks]

(d) By using your results from parts (b) and (c), find the Maclaurin series for $\frac{\mathrm{e}^{2x}-1}{\mathrm{e}^{2x}+1}$ up to and including the term in x^3. [4]

Markscheme

attempt to evaluate at least three of y, y', y'', y''' at $x = 0$ **(M1)**

$y(0) = 0, y'(0) = 1, y''(0) = 0, y'''(0) = -2$ **A1**

attempt to use Maclaurin series $y \approx y(0) + y'(0)x + \frac{y''(0)}{2!}x^2 + \frac{y'''(0)}{3!}x^3 + \ldots$ **M1**

$y \approx 0 + 1x + 0x^2 + \frac{(-2)}{3!}x^3 + \ldots$

$y \approx x - \frac{x^3}{3} + \ldots$ **A1**

[4 marks]

6. [Maximum mark: 7] 24M.1.AHL.TZ1.8

(a) Find the first two non-zero terms in the Maclaurin series of

(a.i) $\sin\left(x^2\right)$; [2]

Markscheme

$\sin\left(x^2\right) = x^2 - \frac{\left(x^2\right)^3}{3!} + \ldots$

$\sin\left(x^2\right) = x^2 - \frac{x^6}{3!} + \ldots \left(= x^2 - \frac{x^6}{6} + \ldots\right)$ ***A1A1***

Note: Award ***A1*** for each term.

[2 marks]

(a.ii) $\sin^2\left(x^2\right)$. [3]

Markscheme

METHOD 1

attempt to square their series for $\sin\left(x^2\right)$ ***(M1)***

$\left(\sin\left(x^2\right)\right)^2 = \left(x^2 - \frac{x^6}{3!} + \ldots\right)^2$

Note: Award ***M0*** for $\left(x^2\right)^2 - \left(\frac{x^6}{3!}\right)^2 + \ldots$.

$= x^4 - \frac{2x^8}{3!} + \ldots \left(x^4 - \frac{x^8}{3} + \ldots\right)$ ***A1A1***

Note: Award ***A1*** for each term.

METHOD 2

attempt to use the identity $\sin^2\left(x^2\right) = \frac{1-\cos\left(2x^2\right)}{2}$ ***(M1)***

$\sin^2\left(x^2\right) = \frac{1}{2}\left(1 - \left(1 - \frac{\left(2x^2\right)^2}{2!} + \frac{\left(2x^2\right)^4}{4!}\right)\right)$

$= x^4 - \frac{8x^8}{4!} + \ldots \left(= x^4 - \frac{x^8}{3} + \ldots\right)$ ***A1A1***

Note: Award ***A1*** for each term.

[3 marks]

(b) Hence, or otherwise, find the first two non-zero terms in the Maclaurin series of $4x \sin\left(x^2\right) \cos\left(x^2\right)$. [2]

Markscheme

METHOD 1

recognition that $4x \sin\left(x^2\right) \cos\left(x^2\right) = \frac{\mathrm{d}\left(\left(\sin\left(x^2\right)\right)^2\right)}{\mathrm{d}x}$ ***(M1)***

$= 4x^3 - \frac{8x^7}{3} + \ldots$ ***A1***

METHOD 2

recognition that $4x \sin\left(x^2\right) \cos\left(x^2\right) = 2x \sin\left(2x^2\right)$ ***(M1)***

$= 2x\left(2x^2 - \frac{\left(2x^2\right)^3}{3!} + \ldots\right)$

$= 4x^3 - \frac{8x^7}{3} + \ldots$ ***A1***

METHOD 3

$4x \sin\left(x^2\right) \cos\left(x^2\right)$

$= 4x\left(x^2 - \frac{x^6}{3!} + \ldots\right)\left(1 - \frac{x^4}{2!} + \ldots\right)$ ***(A1)***

$= 4x^3 - \frac{8x^7}{3} + \ldots$ ***A1***

METHOD 4

recognition that $2x \cos\left(x^2\right) = \frac{\mathrm{d}\left(\sin\left(x^2\right)\right)}{\mathrm{d}x}$ ***(M1)***

$4x \sin\left(x^2\right) \cos\left(x^2\right)$

$= 2\left(x^2 - \frac{x^6}{3!} + \ldots\right)\left(2x - \frac{6x^5}{2!} + \ldots\right)$

$= 4x^3 - \frac{8x^7}{3} + \ldots$ ***A1***

[2 marks]

7. [Maximum mark: 20] 24M.1.AHL.TZ2.12

(a) Let $f(x) = (1 - ax)^{-\frac{1}{2}}$, where $ax < 1,\ a \neq 0$.

The n^{th} derivative of $f(x)$ is denoted by $f^{(n)}(x),\ n \in \mathbb{Z}^+$.

Prove by induction that $f^{(n)}(x) = \frac{a^n(2n-1)!(1-ax)^{-\frac{2n+1}{2}}}{2^{2n-1}(n-1)!},\ n \in \mathbb{Z}^+$. [8]

Markscheme

$n = 1 : LHS = f^{(1)}(x) = -\frac{1}{2} \times -a(1-ax)^{-\frac{3}{2}} \left(= \frac{a}{2}(1-ax)^{-\frac{3}{2}}\right)$ ***A1***

$RHS = \frac{a(1)!(1-ax)^{-\frac{3}{2}}}{2^1(0)!}$ therefore true for $n = 1$ ***R1***

assume (that the result is) true for $n = k$ ***M1***

Note: Do not award ***M1*** for statements such as "let $n = k$" or "assume that $n = k$ is true". The assumption of truth must be clear.

$f^{(k)}(x) = \frac{a^k(2k-1)!(1-ax)^{\frac{(2k+1)}{2}}}{2^{2k-1}(k-1)!}$

attempt to differentiate the right-hand side with respect to x : ***M1***

$f^{(k+1)}(x) = \frac{\mathrm{d}}{\mathrm{d}x}\left(f^{(k)}(x)\right)$

$= \frac{-(2k+1)\times -a}{2} \times \frac{a^k(2k-1)!(1-ax)^{-\frac{(2k+1)}{2}-1}}{2^{2k-1}(k-1)!}$ (or equivalent) ***A1***

attempt to multiply top and bottom by $2k$ ***M1***

$= \frac{(2k+1)}{2} \times \frac{a^{k+1}(2k-1)!(1-ax)^{-\frac{2k+3}{2}}}{2^{2k-1}(k-1)!} \times \frac{2k}{2k}$

$= \frac{a^{k+1}(2k+1)!(1-ax)^{-\frac{2k+3}{2}}}{2^{2k+1}(k)!}$ ***A1***

hence if the result is true for $n = k$ then it is true for $n = k+1$ and as it is true for $n = 1$ it is true for all $n \in \mathbb{Z}^+$ ***R1***

Note: To obtain the final ***R1***, at least five of the previous marks must have been awarded.

[8 marks]

(b) By using part (a) or otherwise, show that the Maclaurin series for $f(x) = (1-ax)^{-\frac{1}{2}}$ up to and including the x^2 term is $1 + \frac{1}{2}ax + \frac{3}{8}a^2x^2$. [2]

Markscheme

$f(x) = f(0) + xf\prime(0) + \frac{x^2}{2}f\prime\prime(0) + \ldots$

$f\prime\prime(x) = \frac{a^2(3)!(1-ax)^{-\frac{5}{2}}}{2^3(1)!}$ OR $\frac{3}{4}a^2(1-ax)^{-\frac{5}{2}}$ OR $f\prime\prime(0) = \frac{a^2(3)!}{2^3}$ **A1**

$f(0) = 1,\ f\prime(0) = \frac{a}{2},\ f\prime\prime(0) = \frac{6a^2}{8}$ **A1**

$f(x) = 1 + \frac{1}{2}ax + \frac{3}{8}a^2x^2 + \ldots$ **AG**

[2 marks]

(c) Hence, show that $(1-2x)^{-\frac{1}{2}}(1-4x)^{-\frac{1}{2}} \approx \frac{2+6x+19x^2}{2}$. [4]

Markscheme

attempt to use $a = 2$ or $a = 4$ in the expansion **M1**

$(1-2x)^{-\frac{1}{2}} = 1 + \frac{2x}{2} + \frac{3\times4x^2}{8}\left(= 1 + x + \frac{3}{2}x^2 + \ldots\right)$

$(1-4x)^{-\frac{1}{2}} = 1 + \frac{4x}{2} + \frac{3\times16x^2}{8}\left(= 1 + 2x + 6x^2 + \ldots\right)$ **A1**

Note: Award ***A1*** for at least one correct.

attempt to multiply their two expansions together **M1**

$\left(1 + x + \frac{3}{2}x^2 + \ldots\right)\left(1 + 2x + 6x^2 + \ldots\right) = 1 + 2x + 6x^2 + x + 2x^2 + \frac{3}{2}x^2 + \ldots$

$1 + 3x + \frac{19}{2}x^2 + \ldots$ OR $\frac{2+4x+12x^2+2x+4x^2+3x^2+\ldots}{2}$ **A1**

$(1-2x)^{-\frac{1}{2}}(1-4x)^{-\frac{1}{2}} \approx \frac{2+6x+19x^2}{2}$ **AG**

[4 marks]

(d) Given that the series expansion for $(1-ax)^{-\frac{1}{2}}$ is convergent for $|ax| < 1$, state the restriction which must be placed on x for the approximation $(1-2x)^{-\frac{1}{2}}(1-4x)^{-\frac{1}{2}} \approx \frac{2+6x+19x^2}{2}$ to be valid. [1]

Markscheme

$|x| < \frac{1}{4}$ **A1**

[1 mark]

(e) Use $x = \frac{1}{10}$ to determine an approximate value for $\sqrt{3}$.

Give your answer in the form $\frac{c}{d}$, where $c,\ d \in \mathbb{Z}^+$. [5]

Markscheme

$\left(\frac{2+6x+19x^2}{2}\right) = \frac{2+0.6+0.19}{2}\left(= \frac{279}{200}\right)$ (or equivalent) ***A1***

attempt to substitute $x = \frac{1}{10}$ into $(1-2x)^{-\frac{1}{2}}(1-4x)^{-\frac{1}{2}}$ ***(M1)***

$\left((1-2x)^{-\frac{1}{2}}(1-4x)^{-\frac{1}{2}} =\right) \frac{10}{\sqrt{48}}$ (or equivalent) ***A1***

$\frac{10}{4\sqrt{3}} \approx \frac{279}{200}$ (or equivalent in terms of $\sqrt{3}$) ***A1***

$\frac{1}{\sqrt{3}} \approx \frac{279}{500}$

$\sqrt{3} \approx \frac{500}{279}$ ***A1***

[5 marks]

8. [Maximum mark: 21] 23N.1.AHL.TZ1.11

Consider the function $f(x) = \mathrm{e}^{\cos 2x}$, where $-\frac{\pi}{4} \leq x \leq \frac{5\pi}{4}$.

(a) Find the coordinates of the points on the curve $y = f(x)$ where the gradient is zero. [5]

Markscheme

attempt to use chain rule to find $f'(x)$ ***(M1)***

$f'(x) = (-2\ \sin\ 2x)\mathrm{e}^{\cos 2x}(= 0)$ ***A1***

$\Rightarrow \sin\ 2x = 0$ ***(M1)***

$2x = 0, \pi, 2\pi,\ \ldots$

$x = 0, \frac{\pi}{2}, \pi,\ \ldots$ ***A1***

Coordinates are $(0, \mathrm{e}),\ \left(\frac{\pi}{2},\ \frac{1}{\mathrm{e}}\right),\ (\pi, \mathrm{e})$ ***A1***

Note: Special case: For two correct coordinate pairs award ***(M1)A1(M1)A0A1***. For extra coordinate pairs award ***(M1)A1(M1)A1A0***.

[5 marks]

(b) Using the second derivative at each point found in part (a), show that the curve $y = f(x)$ has two local maximum points and one local minimum point. [4]

Markscheme

attempt to differentiate $f\prime(x)$ using product rule ***(M1)***

$f\prime\prime(x) = (-2\sin 2x)(-2\sin 2x)\mathrm{e}^{\cos 2x} - (4\cos 2x)\mathrm{e}^{\cos 2x}$ ***A1***

at $x = 0,\ f\prime\prime(x) = -4\mathrm{e} < 0$ so maximum **AND**

at $x = \pi,\ f\prime\prime(\mathrm{x}) = -4\mathrm{e} < 0$ so maximum ***R1***

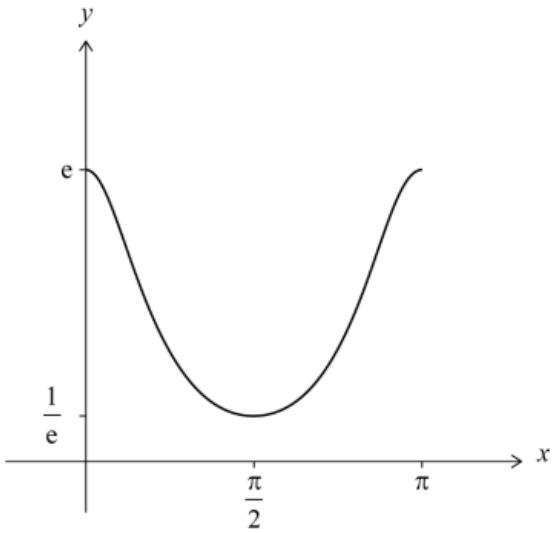
at $x = \frac{\pi}{2},\ f\prime\prime(x) = \frac{4}{\mathrm{e}} > 0$ so minimum ***R1***

Note: The values for the second derivative must be correct in order to award the ***R*** marks.

[4 marks]

(c) Sketch the curve of $y = f(x)$ for $0 \leq x \leq \pi$, taking into consideration the relative values of the second derivative found in part (b). [3]

Markscheme



A1A1A1

Note: Award ***A1*** for general shape, ***A1*** for correct maxima $(0, \mathrm{e}), (\pi, \mathrm{e})$ and minimum point $\left(\frac{\pi}{2},\ \frac{1}{\mathrm{e}}\right)$ and ***A1*** for showing a higher rate of change of gradient at maxima and a lower rate of change of gradient at the minimum point.

[3 marks]

(d.i) Find the Maclaurin series for $\cos 2x$, up to and including the term in x^4. [2]

Markscheme

$\cos 2x = 1 - \frac{(2x)^2}{2!} + \frac{(2x)^4}{4!} + \ldots$ ***(M1)***

$\cos 2x = 1 - 2x^2 + \frac{2x^4}{3} \ldots$ ***A1***

[2 marks]

(d.ii) Hence, find the Maclaurin series for $e^{\cos^{2x-1}}$, up to and including the term in x^4. [3]

Markscheme

METHOD 1

$e^x = 1 + x + \frac{x^2}{2} + \frac{x^3}{6} + \frac{x^4}{24} + \ldots$

attempt to substitute series for $\cos 2x - 1$into series for e^x ***(M1)***

Note: Award **(M0)** for substituting the Maclaurin series for $\cos 2x$ into the Maclaurin series for e^x.

$e^{\cos 2x - 1} = 1 + \left(-2x^2 + \frac{2x^4}{3}\right) + \frac{\left(-2x^2 + \frac{2x^4}{3}\right)^2}{2} + \ldots$ ***A1***

$\left(= 1 - 2x^2 + \frac{2x^4}{3} + 2x^4 + \ldots\right)$

$= 1 - 2x^2 + \frac{8x^4}{3} + \ldots$ ***A1***

METHOD 2

$e^{\cos 2x - 1} = e^{-2x^2 + \frac{2x^4}{3}} = e^{-2x^2} e^{\frac{2x^4}{3}}$

attempt to find the Maclaurin series for e^{-2x^2} OR $e^{\frac{2x^4}{3}}$ ***(M1)***

$e^{-2x^2} = 1 - 2x^2 + 2x^4 + \ldots;\ e^{\frac{2x^4}{3}} = 1 + \frac{2}{3}x^4 \ldots$

$e^{-2x^2} e^{\frac{2}{3}x^4} = \left(1 - 2x^2 + 2x^4 + \ldots\right)\left(1 + \frac{2}{3}x^4 + \ldots\right)$ ***A1***

$= 1 - 2x^2 + \frac{8x^4}{3} + \ldots$ ***A1***

[3 marks]

(d.iii) Hence, write down the Maclaurin series for $f(x)$, up to and including the term in x^4. [1]

Markscheme

$(f(x) \approx)\mathrm{e}\left[1 - 2x^2 + \frac{8x^4}{3} + \ldots\right] \left(= \mathrm{e} - 2\mathrm{e}x^2 + \frac{8\mathrm{e}x^4}{3} + \ldots\right)$ **A1**

[1 mark]

(e) Use the first two non-zero terms in the Maclaurin series for $f(x)$ to show that $\int_0^{1/10} \mathrm{e}^{\cos 2x} \mathrm{d}x \approx \frac{149\mathrm{e}}{1500}$ [3]

Markscheme

$\int_0^{\frac{1}{10}} \mathrm{e}^{\cos 2x} \mathrm{d}x \approx \mathrm{e} \int_0^{\frac{1}{10}} \left(1 - 2x^2\right) \mathrm{d}x$ ***(M1)***

$= \mathrm{e}\left[x - \frac{2x^3}{3}\right]_0^{1/10}$ ***A1***

$= \mathrm{e}\left(\frac{1}{10} - \frac{2}{3000}\right)$ ***A1***

$= \frac{298\mathrm{e}}{3000}$

$= \frac{149\mathrm{e}}{1500}$ ***AG***

Note: If candidate follows through an incorrect expansion from part (d), award a maximum of ***M1A1FTA0***

[3 marks]

9. [Maximum mark: 21] 23N.1.AHL.TZ2.11

Consider the function $f(x) = \mathrm{e}^{\cos 2x}$, where $-\frac{\pi}{4} \leq x \leq \frac{5\pi}{4}$.

(a) Find the coordinates of the points on the curve $y = f(x)$ where the gradient is zero. [5]

Markscheme

attempt to use chain rule to find $f'(x)$ ***(M1)***

$f'(x) = (-2 \sin 2x)\mathrm{e}^{\cos 2x} (= 0)$ ***A1***

$\Rightarrow \sin 2x = 0$ ***(M1)***

$2x = 0, \pi, 2\pi, \ldots$

$x = 0, \frac{\pi}{2}, \pi, \ldots$ ***A1***

Coordinates are $(0, \mathrm{e}), \left(\frac{\pi}{2}, \frac{1}{\mathrm{e}}\right), (\pi, \mathrm{e})$ ***A1***

Note: Special case: For two correct coordinate pairs award ***(M1)A1(M1)A0A1***. For extra coordinate pairs award ***(M1)A1(M1)A1A0***.

[5 marks]

(b) Using the second derivative at each point found in part (a), show that the curve $y = f(x)$ has two local maximum points and one local minimum point. [4]

Markscheme

attempt to differentiate $f\prime(x)$ using product rule ***(M1)***

$f\prime\prime(x) = (-2\sin 2x)(-2\sin 2x)\mathrm{e}^{\cos 2x} - (4\cos 2x)\mathrm{e}^{\cos 2x}$ ***A1***

at $x = 0,\ f\prime\prime(x) = -4\mathrm{e} < 0$ so maximum **AND**

at $x = \pi,\ f\prime\prime(\mathrm{x}) = -4\mathrm{e} < 0$ so maximum ***R1***

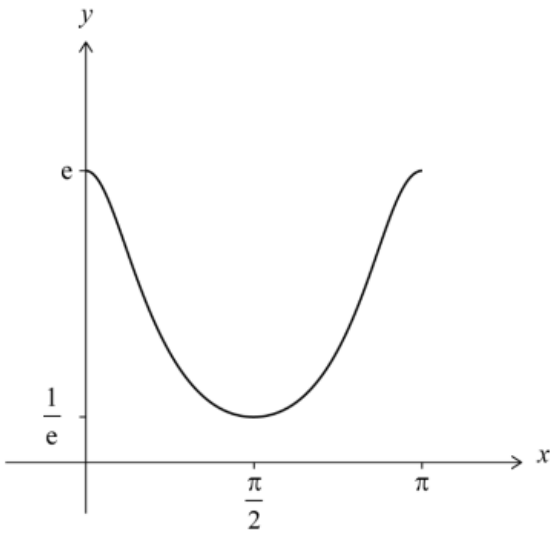
at $x = \frac{\pi}{2},\ f\prime\prime(x) = \frac{4}{\mathrm{e}} > 0$ so minimum ***R1***

Note: The values for the second derivative must be correct in order to award the ***R*** marks.

[4 marks]

(c) Sketch the curve of $y = f(x)$ for $0 \leq x \leq \pi$, taking into consideration the relative values of the second derivative found in part (b). [3]

Markscheme



A1A1A1

Note: Award ***A1*** for general shape, ***A1*** for correct maxima $(0, \mathrm{e}), (\pi, \mathrm{e})$ and minimum point $\left(\frac{\pi}{2}, \frac{1}{\mathrm{e}}\right)$ and ***A1*** for showing a higher rate of change of gradient at maxima and a lower rate of change of gradient at the minimum point.

[3 marks]

(d.i) Find the Maclaurin series for $\cos 2x$, up to and including the term in x^4. [2]

Markscheme

$\cos 2x = 1 - \frac{(2x)^2}{2!} + \frac{(2x)^4}{4!} + \ldots$ ***(M1)***

$\cos 2x = 1 - 2x^2 + \frac{2x^4}{3} \ldots$ ***A1***

[2 marks]

(d.ii) Hence, find the Maclaurin series for $\mathrm{e}^{\cos^{2x-1}}$, up to and including the term in x^4. [3]

Markscheme

METHOD 1

$\mathrm{e}^x = 1 + x + \frac{x^2}{2} + \frac{x^3}{6} + \frac{x^4}{24} + \ldots$

attempt to substitute series for $\cos 2x - 1$into series for e^x ***(M1)***

Note: Award **(M0)** for substituting the Maclaurin series for $\cos 2x$ into the Maclaurin series for e^x.

$\mathrm{e}^{\cos 2x-1} = 1 + \left(-2x^2 + \frac{2x^4}{3}\right) + \frac{\left(-2x^2+\frac{2x^4}{3}\right)^2}{2} + \ldots$ ***A1***

$\left(= 1 - 2x^2 + \frac{2x^4}{3} + 2x^4 + \ldots\right)$

$= 1 - 2x^2 + \frac{8x^4}{3} + \ldots$ ***A1***

METHOD 2

$\mathrm{e}^{\cos 2x-1} = \mathrm{e}^{-2x^2+\frac{2x^4}{3}} = \mathrm{e}^{-2x^2}\mathrm{e}^{\frac{2x^4}{3}}$

attempt to find the Maclaurin series for e^{-2x^2} OR $\mathrm{e}^{\frac{2x^4}{3}}$ ***(M1)***

$\mathrm{e}^{-2x^2} = 1 - 2x^2 + 2x^4 + \ldots;\ \mathrm{e}^{\frac{2x^4}{3}} = 1 + \frac{2}{3}x^4 \ldots$

$\mathrm{e}^{-2x^2}\mathrm{e}^{\frac{2}{3}x^4} = \left(1 - 2x^2 + 2x^4 + \ldots\right)\left(1 + \frac{2}{3}x^4 + \ldots\right)$ ***A1***

$= 1 - 2x^2 + \frac{8x^4}{3} + \ldots$ ***A1***

[3 marks]

(d.iii) Hence, write down the Maclaurin series for $f(x)$, up to and including the term in x^4. [1]

Markscheme

$(f(x) \approx)\mathrm{e}\left[1 - 2x^2 + \frac{8x^4}{3} + \ldots\right] \left(= \mathrm{e} - 2\mathrm{e}x^2 + \frac{8\mathrm{e}x^4}{3} + \ldots\right)$ ***A1***

[1 mark]

(e) Use the first two non-zero terms in the Maclaurin series for $f(x)$ to show that $\int_0^{1/10} \mathrm{e}^{\cos 2x}\mathrm{d}x \approx \frac{149\mathrm{e}}{1500}$ [3]

Markscheme

$\int_0^{\frac{1}{10}} \mathrm{e}^{\cos 2x}\mathrm{d}x \approx \mathrm{e}\int_0^{\frac{1}{10}} \left(1 - 2x^2\right)\mathrm{d}x$ ***(M1)***

$= \mathrm{e}\left[x - \frac{2x^3}{3}\right]_0^{1/10}$ ***A1***

$= \mathrm{e}\left(\frac{1}{10} - \frac{2}{3000}\right)$ ***A1***

$= \frac{298\mathrm{e}}{3000}$

$= \frac{149\mathrm{e}}{1500}$ ***AG***

Note: If candidate follows through an incorrect expansion from part (d), award a maximum of ***M1A1FTA0***

[3 marks]

10. [Maximum mark: 21] 22M.1.AHL.TZ1.12

The function f is defined by $f(x) = \mathrm{e}^x \sin x$, where $x \in \mathbb{R}$.

(a) Find the Maclaurin series for $f(x)$ up to and including the x^3 term. [4]

Markscheme

METHOD 1

recognition of both known series ***(M1)***

$\mathrm{e}^x = 1 + \frac{x}{1!} + \frac{x^2}{2!} + \ldots$ and $\sin x = x - \frac{x^3}{3!} + \frac{x^5}{5!} + \ldots$

attempt to multiply the two series up to and including x^3 term ***(M1)***

$e^x \sin x = \left(1 + \frac{x}{1!} + \frac{x^2}{2!} + \ldots\right)\left(x - \frac{x^3}{3!} + \frac{x^5}{5!} + \ldots\right)$

$= x - \frac{x^3}{3!} + x^2 + \frac{x^3}{2!} + \ldots$ ***(A1)***

$e^x \sin x = x + x^2 + \frac{1}{3}x^3 + \ldots$ ***A1***

METHOD 2

$f(x) = e^x \sin x$

$f'(x) = e^x \cos x + e^x \sin x$ ***A1***

$f''(x) = e^x \cos x - e^x \sin x + e^x \sin x + e^x \cos x \ (= 2e^x \cos x)$

$f'''(x) = 2e^x \cos x - 2e^x \sin x$

$f''(x) = 2e^x \cos x$ and $f'''(x) = 2e^x(\cos x - \sin x)$ ***A1***

substitute $x = 0$ into f or its derivatives to obtain Maclaurin series ***(M1)***

$e^x \sin x = 0 + \frac{x}{1!} \times 1 + \frac{x^2}{2!} \times 2 + \frac{x^3}{3!} \times 2 + \ldots$

$e^x \sin x = x + x^2 + \frac{1}{3}x^3 + \ldots$ ***A1***

[4 marks]

(b) Hence, find an approximate value for $\int_0^1 e^{x^2} \sin\left(x^2\right) \mathrm{d}x$. [4]

Markscheme

$e^{x^2} \sin\left(x^2\right) = x^2 + x^4 + \frac{1}{3}x^6 + \ldots$ ***(A1)***

substituting their expression and attempt to integrate ***M1***

$\int_0^1 e^{x^2} \sin\left(x^2\right) \mathrm{d}x \approx \int_0^1 \left(x^2 + x^4 + \frac{1}{3}x^6\right) \mathrm{d}x$

Note: Condone absence of limits up to this stage.

$= \left[\frac{x^3}{3} + \frac{x^5}{5} + \frac{x^7}{21}\right]_0^1$ ***A1***

$= \frac{61}{105}$ ***A1***

[4 marks]

The function g is defined by $g(x) = \mathrm{e}^x \cos x$, where $x \in \mathbb{R}$.

(c.i) Show that $g(x)$ satisfies the equation $g''(x) = 2(g'(x) - g(x))$. [4]

Markscheme

attempt to use product rule at least once ***M1***

$g'(x) = \mathrm{e}^x \cos x - \mathrm{e}^x \sin x$ ***A1***

$g''(x) = \mathrm{e}^x \cos x - \mathrm{e}^x \sin x - \mathrm{e}^x \sin x - \mathrm{e}^x \cos x (= -2\mathrm{e}^x \sin x)$ ***A1***

EITHER

$2(g'(x) - g(x)) = 2(\mathrm{e}^x \cos x - \mathrm{e}^x \sin x - \mathrm{e}^x \cos x) = -2\mathrm{e}^x \sin x$ ***A1***

OR

$g''(x) = 2(\mathrm{e}^x \cos x - \mathrm{e}^x \sin x - \mathrm{e}^x \cos x)$ ***A1***

THEN

$g''(x) = 2(g'(x) - g(x))$ ***AG***

Note: Accept working with each side separately to obtain $-2\mathrm{e}^x \sin x$.

[4 marks]

(c.ii) Hence, deduce that $g^{(4)}(x) = 2(g'''(x) - g''(x))$. [1]

Markscheme

$g'''(x) = 2(g''(x) - g'(x))$ ***A1***

$g^{(4)}(x) = 2(g'''(x) - g''(x))$ ***AG***

Note: Accept working with each side separately to obtain $-4\mathrm{e}^x \cos x$.

[1 mark]

(d) Using the result from part (c), find the Maclaurin series for $g(x)$ up to and including the x^4 term. [5]

Markscheme

attempt to substitute $x = 0$ into a derivative ***(M1)***

$g(0) = 1,\ g\prime(0) = 1,\ g\prime\prime(0) = 0$ ***A1***

$g\prime\prime\prime(0) = -2,\ g^{(4)}(0) = -4$ ***(A1)***

attempt to substitute into Maclaurin formula ***(M1)***

$g(x) = 1 + x - \frac{2}{3!}x^3 - \frac{4}{4!}x^4 + \ldots \left(= 1 + x - \frac{1}{3}x^3 - \frac{1}{6}x^4 + \ldots\right)$ ***A1***

Note: Do not award any marks for approaches that do not use the part (c) result.

[5 marks]

(e) Hence, or otherwise, determine the value of $\lim_{x \to 0} \frac{e^x \cos x - 1 - x}{x^3}$. [3]

Markscheme

METHOD 1

$\lim_{x \to 0} \frac{e^x \cos x - 1 - x}{x^3} = \lim_{x \to 0} \frac{\left(1 + x - \frac{1}{3}x^3 - \frac{1}{6}x^4 + \ldots\right) - 1 - x}{x^3}$ ***M1***

$= \lim_{x \to 0}\left(-\frac{1}{3} - \frac{1}{6}x + \ldots\right)$ ***(A1)***

$= -\frac{1}{3}$ ***A1***

Note: Condone the omission of $+ \ldots$ in their working.

METHOD 2

$\lim_{x \to 0} \frac{e^x \cos x - 1 - x}{x^3} = \frac{0}{0}$ indeterminate form, attempt to apply l'Hôpital's rule ***M1***

$= \lim_{x \to 0} \frac{e^x \cos x - e^x \sin x - 1}{3x^2} \left(= \lim_{x \to 0} \frac{g\prime(x) - 1}{3x^2}\right)$

$= \frac{0}{0}$, using l'Hôpital's rule again

$= \lim_{x \to 0} \frac{-2e^x \sin x}{6x} \left(= \lim_{x \to 0} \frac{g''(x)}{6x} \right)$

$= \frac{0}{0}$, using l'Hôpital's rule again

$= \lim_{x \to 0} \frac{-2e^x \sin x - 2e^x \cos x}{6} \left(= \lim_{x \to 0} \frac{g'''(x)}{6} \right)$ **A1**

$= -\frac{1}{3}$ **A1**

[3 marks]

11. [Maximum mark: 14] 21N.1.AHL.TZ0.11

(a) Prove by mathematical induction that $\frac{d^n}{dx^n}\left(x^2 e^x\right) = \left[x^2 + 2nx + n(n-1)\right]e^x$ for $n \in \mathbb{Z}^+$. [7]

Markscheme

For $n = 1$

LHS: $\frac{d}{dx}\left(x^2 e^x\right) = x^2 e^x + 2x e^x \left(= e^x\left(x^2 + 2x\right)\right)$ **A1**

RHS: $\left(x^2 + 2(1)x + 1(1-1)\right)e^x \left(= e^x\left(x^2 + 2x\right)\right)$ **A1**

so true for $n = 1$

now assume true for $n = k$; i.e. $\frac{d^k}{dx^k}\left(x^2 e^x\right) = \left[x^2 + 2kx + k(k-1)\right]e^x$ **M1**

Note: Do not award ***M1*** for statements such as "let $n = k$". Subsequent marks can still be awarded.

attempt to differentiate the RHS **M1**

$\frac{d^{k+1}}{dx^{k+1}}\left(x^2 e^x\right) = \frac{d}{dx}\left(\left[x^2 + 2kx + k(k-1)\right]e^x\right)$

$= (2x + 2k)e^x + \left(x^2 + 2kx + k(k-1)\right)e^x$ **A1**

$= \left[x^2 + 2(k+1)x + k(k+1)\right]e^x$ **A1**

so true for $n = k$ implies true for $n = k + 1$

therefore $n = 1$ true and $n = k$ true $\Rightarrow n = k + 1$ true

therefore, true for all $n \in \mathbb{Z}^+$ **R1**

Note: Award ***R1*** only if three of the previous four marks have been awarded

[7 marks]

(b) Hence or otherwise, determine the Maclaurin series of $f(x) = x^2 e^x$ in ascending powers of x, up to and including the term in x^4. [3]

Markscheme

METHOD 1

attempt to use $\frac{d^n}{dx^n}\left(x^2 e^x\right) = \left[x^2 + 2nx + n(n-1)\right]e^x$ ***(M1)***

Note: For $x = 0$, $\frac{d^n}{dx^n}\left(x^2 e^x\right)_{|x=0} = n(n-1)$ may be seen.

$f(0) = 0, \quad f\prime(0) = 0, \quad f\prime\prime(0) = 2, \quad f\prime\prime\prime(0) = 6, \quad f^{(4)}(0) = 12$

use of $f(x) = f(0) + xf\prime(0) + \frac{x^2}{2!}f\prime\prime(0) + \frac{x^3}{3!}f\prime\prime\prime(0) + \frac{x^4}{4!}f^{(4)}(0) + \ldots$ ***(M1)***

$\Rightarrow f(x) \approx x^2 + x^3 + \frac{1}{2}x^4$ ***A1***

METHOD 2

'$x^2 \times$ Maclaurin series of e^x' ***(M1)***

$x^2\left(1 + x + \frac{x^2}{2!} + \ldots\right)$ ***(A1)***

$\Rightarrow f(x) \approx x^2 + x^3 + \frac{1}{2}x^4$ ***A1***

[3 marks]

(c) Hence or otherwise, determine the value of $\lim_{x \to 0}\left[\frac{\left(x^2 e^x - x^2\right)^3}{x^9}\right]$. [4]

Markscheme

METHOD 1

attempt to substitute $x^2 e^x \approx x^2 + x^3 + \frac{1}{2}x^4$ into $\frac{\left(x^2 e^x - x^2\right)^3}{x^9}$ ***M1***

$\frac{\left(x^2 e^x - x^2\right)^3}{x^9} \approx \frac{\left(x^2 + x^3 + \frac{1}{2}x^4(+\ldots) - x^2\right)^3}{x^9}$ ***(A1)***

EITHER

$= \frac{\left(x^3+\frac{1}{2}x^4+\ldots\right)^3}{x^9}$ **A1**

$= \frac{x^9(+\text{ higher order terms})}{x^9}$

OR

$\left(\frac{x^3+\frac{1}{2}x^4(+\ldots)}{x^3}\right)^3$ **A1**

$\left(1+\frac{1}{2}x(+\ldots)\right)^3$

THEN

$= 1\ (+\text{ higher order terms})$

so $\lim_{x\to 0}\left[\frac{\left(x^2e^x-x^2\right)^3}{x^9}\right] = 1$ **A1**

METHOD 2

$\lim_{x\to 0}\left[\frac{\left(x^2e^x-x^2\right)^3}{x^9}\right] = \lim_{x\to 0}\left(\frac{x^2e^x-x^2}{x^3}\right)^3$ **M1**

$= \lim_{x\to 0}\left(\frac{e^x-1}{x}\right)^3$ **(A1)**

attempt to use L'Hôpital's rule **M1**

$= \lim_{x\to 0}\left(\frac{e^x-0}{1}\right)^3$

$= \left[\lim_{x\to 0} e^x\right]^3$

$= 1$ **A1**

[4 marks]

12. [Maximum mark: 20] 21M.1.AHL.TZ1.12

Let $f(x) = \sqrt{1+x}$ for $x > -1$.

(a) Show that $f''(x) = -\frac{1}{4\sqrt{(1+x)^3}}$. [3]

Markscheme

attempt to use the chain rule **M1**

$f\prime(x) = \frac{1}{2}(1+x)^{-\frac{1}{2}}$ **A1**

$f\prime\prime(x) = -\frac{1}{4}(1+x)^{-\frac{3}{2}}$ **A1**

$= -\frac{1}{4\sqrt{(1+x)^3}}$ **AG**

Note: Award **M1A0A0** for $f\prime(x) = \frac{1}{\sqrt{1+x}}$ or equivalent seen

[3 marks]

(b) Use mathematical induction to prove that $f^{(n)}(x) = \left(-\frac{1}{4}\right)^{n-1}\frac{(2n-3)!}{(n-2)!}(1+x)^{\frac{1}{2}-n}$ for $n \in \mathbb{Z},\ n \geq 2$. [9]

Markscheme

let $n = 2$

$f''(x) = \left(-\frac{1}{4\sqrt{(1+x)^3}} = \right)\left(-\frac{1}{4}\right)^1\frac{1!}{0!}(1+x)^{\frac{1}{2}-2}$ **R1**

Note: Award **R0** for not starting at $n = 2$. Award subsequent marks as appropriate.

assume true for $n = k$, (so $f^{(k)}(x) = \left(-\frac{1}{4}\right)^{k-1}\frac{(2k-3)!}{(k-2)!}(1+x)^{\frac{1}{2}-k}$) **M1**

Note: Do not award **M1** for statements such as "let $n = k$" or "$n = k$ is true". Subsequent marks can still be awarded.

consider $n = k+1$

$\text{LHS} = f^{(k+1)}(x) = \frac{d\left(f^{(k)}(x)\right)}{dx}$ **M1**

$= \left(-\frac{1}{4}\right)^{k-1}\frac{(2k-3)!}{(k-2)!}\left(\frac{1}{2}-k\right)(1+x)^{\frac{1}{2}-k-1}$ (or equivalent) **A1**

EITHER

$\text{RHS} = f^{(k+1)}(x) = \left(-\frac{1}{4}\right)^k \frac{(2k-1)!}{(k-1)!}(1+x)^{\frac{1}{2}-k-1}$ (or equivalent) ***A1***

$= \left(-\frac{1}{4}\right)^k \frac{(2k-1)(2k-2)(2k-3)!}{(k-1)(k-2)!}(1+x)^{\frac{1}{2}-k-1}$ ***A1***

Note: Award ***A1*** for $\frac{(2k-1)!}{(k-1)!} = \frac{(2k-1)(2k-2)(2k-3)!}{(k-1)(k-2)!}\left(= \frac{2(2k-1)(2k-3)!}{(k-2)!}\right)$

$= \left(-\frac{1}{4}\right)\left(-\frac{1}{4}\right)^{k-1} \frac{(2k-1)(2k-2)(2k-3)!}{(k-1)(k-2)!}(1+x)^{\frac{1}{2}-k-1}$ ***A1***

$\left(= \left(-\frac{1}{2}\right)\left(-\frac{1}{4}\right)^{k-1} \frac{(2k-1)(2k-3)!}{(k-2)!}(1+x)^{\frac{1}{2}-k-1}\right)$

Note: Award ***A1*** for leading coefficient of $-\frac{1}{4}$.

$= \left(\frac{1}{2}-k\right)\left(-\frac{1}{4}\right)^{k-1} \frac{(2k-3)!}{(k-2)!}(1+x)^{\frac{1}{2}-k-1}$ ***A1***

OR

Note: The following ***A*** marks can be awarded in any order.

$= \left(-\frac{1}{4}\right)^{k-1} \frac{(2k-3)!}{(k-2)!}\left(\frac{1-2k}{2}\right)(1+x)^{\frac{1}{2}-k-1}$

$= \left(-\frac{1}{2}\right)\left(-\frac{1}{4}\right)^{k-1} \frac{(2k-1)(2k-3)!}{(k-2)!}(1+x)^{\frac{1}{2}-k-1}$ ***A1***

Note: Award ***A1*** for isolating $(2k-1)$ correctly.

$= \left(-\frac{1}{2}\right)\left(-\frac{1}{4}\right)^{k-1} \frac{(2k-1)!}{(2k-2)(k-2)!}(1+x)^{\frac{1}{2}-k-1}$ ***A1***

Note: Award ***A1*** for multiplying top and bottom by $(k-1)$ or $2(k-1)$.

$= \left(-\frac{1}{4}\right)\left(-\frac{1}{4}\right)^{k-1} \frac{(2k-1)!}{(k-1)(k-2)!}(1+x)^{\frac{1}{2}-k-1}$ ***A1***

Note: Award ***A1*** for leading coefficient of $-\frac{1}{4}$.

$= \left(-\frac{1}{4}\right)^k \frac{(2k-1)!}{(k-1)!}(1+x)^{\frac{1}{2}-k-1}$ ***A1***

$= \left(-\frac{1}{4}\right)^{(k+1)-1} \frac{(2(k+1)-3)!}{((k+1)-2)!}(1+x)^{\frac{1}{2}-(k+1)} = \text{RHS}$

THEN

since true for $n = 2$, and true for $n = k+1$ if true for $n = k$, the statement is true for all, $n \in \mathbb{Z},\ n \geq 2$ by mathematical induction ***R1***

Note: To obtain the final ***R1***, at least four of the previous marks must have been awarded.

[9 marks]

(c) Let $g(x) = \mathrm{e}^{mx},\ m \in \mathbb{Q}$.

Consider the function h defined by $h(x) = f(x) \times g(x)$ for $x > -1$.

It is given that the x^2 term in the Maclaurin series for $h(x)$ has a coefficient of $\frac{7}{4}$.

Find the possible values of m. [8]

Markscheme

METHOD 1

$h(x) = \sqrt{1+x}\,\mathrm{e}^{mx}$

using product rule to find $h\prime(x)$ ***(M1)***

$h\prime(x) = \sqrt{1+x}\,m\mathrm{e}^{mx} + \frac{1}{2\sqrt{1+x}}\mathrm{e}^{mx}$ ***A1***

$h\prime\prime(x) = m\left(\sqrt{1+x}\,m\mathrm{e}^{mx} + \frac{1}{2\sqrt{1+x}}\mathrm{e}^{mx}\right) + \frac{1}{2\sqrt{1+x}}m\mathrm{e}^{mx} - \frac{1}{4\sqrt{(1+x)^3}}\mathrm{e}^{mx}$ ***A1***

substituting $x = 0$ into $h\prime\prime(x)$ ***M1***

$h\prime\prime(0) = m^2 + \frac{1}{2}m + \frac{1}{2}m - \frac{1}{4}\left(= m^2 + m - \frac{1}{4}\right)$ ***A1***

$h(x) = h(0) + xh\prime(0) + \frac{x^2}{2!}h\prime\prime(0) + \ldots$

equating x^2 coefficient to $\frac{7}{4}$ ***M1***

$\frac{h''(0)}{2!} = \frac{7}{4} \left(\Rightarrow h''(0) = \frac{7}{2}\right)$

$4m^2 + 4m - 15 = 0$ ***A1***

$(2m + 5)(2m - 3) = 0$

$m = -\frac{5}{2}$ or $m = \frac{3}{2}$ ***A1***

METHOD 2

EITHER

attempt to find $f(0),\ f'(0),\ f''(0)$ ***(M1)***

$f(x) = (1 + x)^{\frac{1}{2}}$ $\qquad$ $f(0) = 1$

$f'(x) = \frac{1}{2}(1 + x)^{-\frac{1}{2}}$ $\qquad$ $f'(0) = \frac{1}{2}$

$f''(x) = -\frac{1}{4}(1 + x)^{-\frac{3}{2}}$ $\qquad$ $f''(0) = -\frac{1}{4}$

$f(x) = 1 + \frac{1}{2}x - \frac{1}{8}x^2 + \ldots$ ***A1***

OR

attempt to apply binomial theorem for rational exponents ***(M1)***

$f(x) = (1 + x)^{\frac{1}{2}} = 1 + \frac{1}{2}x + \frac{\left(\frac{1}{2}\right)\left(-\frac{1}{2}\right)}{2!}x^2 \ldots$

$f(x) = 1 + \frac{1}{2}x - \frac{1}{8}x^2 + \ldots$ ***A1***

THEN

$g(x) = 1 + mx + \frac{m^2}{2}x^2 + \ldots$ ***(A1)***

$h(x) = \left(1 + \frac{1}{2}x - \frac{1}{8}x^2 + \ldots\right)\left(1 + mx + \frac{m^2}{2}x^2 + \ldots\right)$ ***(M1)***

coefficient of x^2 is $\frac{m^2}{2} + \frac{m}{2} - \frac{1}{8}$ ***A1***

attempt to set equal to $\frac{7}{4}$ and solve ***M1***

$\frac{m^2}{2} + \frac{m}{2} - \frac{1}{8} = \frac{7}{4}$

$4m^2 + 4m - 15 = 0$ ***A1***

$(2m + 5)(2m - 3) = 0$

$m = -\frac{5}{2}$ or $m = \frac{3}{2}$ ***A1***

METHOD 3

$g\prime(x) = m\mathrm{e}^{mx}$ and $g\prime\prime(x) = m^2\mathrm{e}^{mx}$ ***(A1)***

$h(x) = h(0) + xh\prime(0) + \frac{x^2}{2!}h\prime\prime(0) + \ldots$

equating x^2 coefficient to $\frac{7}{4}$ ***M1***

$\frac{h\prime\prime(0)}{2!} = \frac{7}{4}\ \left(\Rightarrow h\prime\prime(0) = \frac{7}{2}\right)$

using product rule to find $h\prime(x)$ and $h\prime\prime(x)$ ***(M1)***

$h\prime(x) = f(x)g\prime(x) + f\prime(x)g(x)$

$h\prime\prime(x) = f(x)g\prime\prime(x) + 2f\prime(x)g\prime(x) + f\prime\prime(x)g(x)$ ***A1***

substituting $x = 0$ into $h\prime\prime(x)$ ***M1***

$h\prime\prime(0) = f(0)g\prime\prime(0) + 2g\prime(0)f\prime(0) + g(0)f\prime\prime(0)$

$= 1 \times m^2 + 2m \times \frac{1}{2} + 1 \times \left(-\frac{1}{4}\right)\ \left(= m^2 + m - \frac{1}{4}\right)$ ***A1***

$4m^2 + 4m - 15 = 0$ ***A1***

$(2m + 5)(2m - 3) = 0$

$m = -\frac{5}{2}$ or $m = \frac{3}{2}$ ***A1***

[8 marks]

13. [Maximum mark: 8] 21M.2.AHL.TZ2.9

(a) Write down the first three terms of the binomial expansion of $(1 + t)^{-1}$ in ascending powers of t. [1]

Markscheme

$1 - t + t^2$ ***A1***

Note: Accept $1,\ -t$ and t^2.

[1 mark]

(b) By using the Maclaurin series for $\cos x$ and the result from part (a), show that the Maclaurin series for $\sec x$ up to and including the term in x^4 is $1 + \frac{x^2}{2} + \frac{5x^4}{24}$. [4]

Markscheme

$\sec x = \frac{1}{1-\frac{x^2}{2!}+\frac{x^4}{4!}(-\ldots)} \left(= \left(1 - \frac{x^2}{2!} + \left(\frac{x^4}{4!}(-\ldots)\right)\right)^{-1}\right)$ ***(M1)***

$t = \cos x - 1$ or $\sec x = 1 - (\cos x - 1) + (\cos x - 1)^2$ ***(M1)***

$= 1 - \left(-\frac{x^2}{2!} + \frac{x^4}{4!}(-\ldots)\right) + \left(-\frac{x^2}{2!} + \frac{x^4}{4!}(-\ldots)\right)^2$ ***A1***

$= 1 + \frac{x^2}{2} - \frac{x^4}{24} + \frac{x^4}{4}$ ***A1***

so the Maclaurin series for $\sec x$ up to and including the term in x^4 is $1 + \frac{x^2}{2} + \frac{5x^4}{24}$ ***AG***

Note: Condone the absence of '…'

[4 marks]

(c) By using the Maclaurin series for $\arctan x$ and the result from part (b), find $\lim_{x\to 0}\left(\frac{x \arctan 2x}{\sec x - 1}\right)$. [3]

Markscheme

$\arctan 2x = 2x - \frac{(2x)^3}{3} + \ldots$

$\lim_{x\to 0}\left(\frac{x \arctan 2x}{\sec x - 1}\right) = \lim_{x\to 0}\left(\frac{x\left(2x - \frac{(2x)^3}{3} + \ldots\right)}{\left(1 + \frac{x^2}{2} + \frac{5x^4}{24}\right) - 1}\right)$ ***M1***

$= \lim_{x\to 0}\left(\frac{2x^2 - \frac{8x^4}{3} + \ldots}{\frac{x^2}{2} + \frac{5x^4}{24}}\right)$ ***A1***

$= \lim_{x\to 0}\left(\frac{2x^2\left(1 - \frac{4x^2}{3}\right)}{\frac{x^2}{2}\left(1 + \frac{5x^2}{12}\right)}\right)$

$= 4$ ***A1***

Note: Condone missing 'lim' and errors in higher derivatives.
Do not award ***M1*** unless x is replaced by $2x$ in $\arctan$.

[3 marks]

14. [Maximum mark: 21] SPM.1.AHL.TZ0.12

The function f is defined by $f(x) = e^{\sin x}$.

(a) Find the first two derivatives of $f(x)$ and hence find the Maclaurin series for $f(x)$ up to and including the x^2 term. [8]

Markscheme

attempting to use the chain rule to find the first derivative ***M1***

$f'(x) = (\cos x)e^{\sin x}$ ***A1***

attempting to use the product rule to find the second derivative ***M1***

$f''(x) = e^{\sin x}\left(\cos^2 x - \sin x\right)$ (or equivalent) ***A1***

attempting to find $f(0)$, $f'(0)$ and $f''(0)$ **M1**

$f(0) = 1$; $f'(0) = (\cos 0)e^{\sin 0} = 1$; $f''(0) = e^{\sin 0}\left(\cos^2 0 - \sin 0\right) = 1$ ***A1***

substitution into the Maclaurin formula $f(x) = f(0) + xf'(0) + \frac{x^2}{2!}f''(0) + \ldots$ **M1**

so the Maclaurin series for $f(x)$ up to and including the x^2 term is $1 + x + \frac{x^2}{2}$ ***A1***

[8 marks]

(b) Show that the coefficient of x^3 in the Maclaurin series for $f(x)$ is zero. [4]

Markscheme

METHOD 1

attempting to differentiate $f''(x)$ ***M1***

$f'''(x) = (\cos x)e^{\sin x}\left(\cos^2 x - \sin x\right) - (\cos x)e^{\sin x}(2\sin x + 1)$ (or equivalent) ***A2***

substituting $x = 0$ into **their** $f'''(x)$ ***M1***

$f'''(0) = 1(1 - 0) - 1(0 + 1) = 0$

so the coefficient of x^3 in the Maclaurin series for $f(x)$ is zero ***AG***

METHOD 2

substituting $\sin x$ into the Maclaurin series for e^x ***(M1)***

$e^{\sin x} = 1 + \sin x + \frac{\sin^2 x}{2!} + \frac{\sin^3 x}{3!} + \ldots$

substituting Maclaurin series for $\sin x$ ***M1***

$e^{\sin x} = 1 + \left(x - \frac{x^3}{3!} + \ldots\right) + \frac{\left(x - \frac{x^3}{3!} + \ldots\right)^2}{2!} + \frac{\left(x - \frac{x^3}{3!} + \ldots\right)^3}{3!} + \ldots$ ***A1***

coefficient of x^3 is $-\frac{1}{3!} + \frac{1}{3!} = 0$ ***A1***

so the coefficient of x^3 in the Maclaurin series for $f(x)$ is zero ***AG***

[4 marks]

(c) Using the Maclaurin series for $\arctan x$ and $e^{3x} - 1$, find the Maclaurin series for $\arctan\left(e^{3x} - 1\right)$ up to and including the x^3 term. [6]

Markscheme

substituting $3x$ into the Maclaurin series for e^x ***M1***

$e^{3x} = 1 + 3x + \frac{(3x)^2}{2!} + \frac{(3x)^3}{3!} + \ldots$ ***A1***

substituting $\left(e^{3x} - 1\right)$ into the Maclaurin series for $\arctan x$ ***M1***

$\arctan\left(e^{3x} - 1\right) = \left(e^{3x} - 1\right) - \frac{\left(e^{3x} - 1\right)^3}{3} + \frac{\left(e^{3x} - 1\right)^5}{5} - \ldots$

$= \left(3x + \frac{(3x)^2}{2!} + \frac{(3x)^3}{3!} + \ldots\right) - \frac{\left(3x + \frac{(3x)^2}{2!} + \frac{(3x)^3}{3!} + \ldots\right)^3}{3} + \ldots$ ***A1***

selecting correct terms from above ***M1***

$= \left(3x + \frac{(3x)^2}{2!} + \frac{(3x)^3}{3!}\right) - \frac{(3x)^3}{3}$

$= 3x + \frac{9x^2}{2} - \frac{9x^3}{2}$ ***A1***

[6 marks]

(d) Hence, or otherwise, find $\lim_{x \to 0} \frac{f(x) - 1}{\arctan\left(e^{3x} - 1\right)}$. [3]

Markscheme

METHOD 1

substitution of **their** series ***M1***

$\lim_{x \to 0} \frac{x + \frac{x^2}{2} + \ldots}{3x + \frac{9x^2}{2} + \ldots}$ ***A1***

$= \lim_{x \to 0} \frac{1 + \frac{x}{2} + \ldots}{3 + \frac{9x}{2} + \ldots}$

$= \frac{1}{3}$ ***A1***

METHOD 2

use of l'Hôpital's rule ***M1***

$\lim_{x \to 0} \frac{(\cos x)e^{\sin x}}{\frac{3e^{3x}}{1+\left(e^{3x}-1\right)^2}}$ (or equivalent) ***A1***

$= \frac{1}{3}$ ***A1***

[3 marks]

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